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The order of the quantum corrections to the total radiation intensity of the glowing electron is found, and their value is calculated.

In the preceeding work /1/ (quoted further as I), we obtained the expression (see (9), I) for the total radiation intensity, which in the general case can be reduced to the form:

$$W = \frac{ce^2}{4\pi} \sum_{l'=0}^{l} \oint d\Omega \varkappa^2 \sum_{s,s'=\pm 1} (|\bar{\alpha}_x|^2 + \cos^2\theta |\bar{\alpha}_y|^2 + \sin^2\theta |\bar{\alpha}_z|^2 - - (\bar{\alpha}_y^+ \bar{\alpha}_z + \bar{\alpha}_z^+ \bar{\alpha}_y) \sin\theta\cos\theta) \frac{1}{\partial(K' + \varkappa)/\partial\varkappa}, \tag{1}$$

and furthermore, according to formula (16), I

$$\begin{aligned}
\sum_{s,s'=\pm 1} |\bar{\alpha}_x|^2 &= \frac{KK' - k_0^2}{2KK'} (|I(l', l-1)|^2 + |I(l'-1, l)|^2) - \\
&- \frac{\gamma\sqrt{ll'}}{KK'} (I^+(l', l-1) I(l'-1, l) + I^+(l'-1, l) I(l', l-1)), \\
\sum_{s,s'=\pm 1} |\bar{\alpha}_y|^2 &= \frac{KK' - k_0^2}{2KK'} (|I(l', l-1)|^2 + |I(l'-1, l)|^2) + \\
&+ \frac{\gamma\sqrt{ll'}}{KK'} (I^+(l', l-1) I(l'-1, l) + I^+(l'-1, l) I(l', l-1)), \\
\sum_{s,s'=\pm 1} |\bar{\alpha}_z|^2 &= \frac{KK' - k_0^2}{2KK'} (|I(l', l)|^2 + |I(l'-1, l-1)|^2) - \\
&- \frac{\gamma\sqrt{ll'}}{KK'} (I^+(l', l) I(l'-1, l-1) + I^+(l'-1, l-1) I(l', l)), \\
\sum_{s,s'=\pm 1} (\bar{\alpha}_y^+ \bar{\alpha}_z + \bar{\alpha}_z^+ \bar{\alpha}_y) &= -2 \frac{\varkappa\sqrt{2\gamma l}\cos\theta}{2KK'} (I^+(l'-1, l) I(l'-1, l-1) + \\
&+ I^+(l', l-1) I(l', l)).
\end{aligned} \tag{2}$$

The value I $I(l', l)$ is in our case connected with the Sonin-Laguerre polynomial by the relationship

$$I^*(l', l) = I(l', l) = \frac{1}{\sqrt{n\, l!}} x^{(l-l')/2} e^{-x/2} Q_{l'}^{(l-l')}(x), \qquad (3)$$

where

$$x = \frac{l}{\beta^2 \sin^2\theta}\left(1 - \sqrt{1 - \frac{n}{l}\beta^2 \sin^2\theta}\right)^2. \qquad (4)$$

Disregarding values of the order of n^2/l^2 and n^a/l $(a < 1)$, we shall find, taking into account formulas (20) - (22) of work I

$$
\left.
\begin{aligned}
|\overline{\alpha}_x|^2 &= \frac{\beta^2}{2}\left(I(l', l-1) - I(l'-1, l)\right)^2, \\
|\overline{\alpha}_y|^2 &= \frac{\beta^2}{2}\left(I(l', l-1) + I(l'-1, l)\right)^2, \\
|\overline{\alpha}_z|^2 &= \frac{\beta^2}{2}\left(I(l', l) - I(l'-1, l-1)\right)^2, \\
\overline{\alpha}_y^{+}\overline{\alpha}_z + \overline{\alpha}_z^{+}\overline{\alpha}_y &= \frac{\beta^2}{2}\left(I(l', l)\, I(l', l-1) + I(l'-1, l-1)\, I(l'-1, l)\right).
\end{aligned}
\right\} \qquad (5)
$$

In work I we were able to calculate W with the assumption that $n^3/l \ll 1$. We showed that in this case, when the total radiation intensity is calculated, terms of the order of n^2/l contract, and the intensity then depends only on terms of the order of n/l.

In the present work, developing the method of /1,2/, we shall find the total radiation intensity, proceeding only from the assumption that $n/l \ll 1$. With this purpose in mind, we shall find the asymptotic approximation of matrix element I (l', l) for values of x which correspond to the radiation maximum $\left(\theta \sim \frac{\pi}{2},\ x \sim \frac{n^2}{4l}\right)$. As is known, function I (l', l) satisfies this differential equation (see, for example, /2/)

$$\frac{d^2}{dx^2}(\sqrt{x}\, I(l',l)) + \left(-\frac{1}{4} + \frac{l+l'+1}{2x} - \frac{(l-l')^2-1}{4x^2}\right)\sqrt{x}\, I(l',l) = 0, \quad (6)$$

moreover, with $x \to 0$ the solution becomes equal to

$$I(l',l) = \sqrt{\frac{l!}{l'!}}\, x^{(l-l')/2} \frac{1}{(l-l')!}.$$

According to /4/ an asymptotic solution of equation (6), equally well applicable in the region of interest to us $x \to x_0$ $(x < x_0)$, as well as when $x \to 0$ $(x > 0)$, can be represented in the form:

$$I(l',l) = A\sqrt{-\frac{z}{z'x}}\, K_{1/3}(z), \quad (7)$$

where

$$z = \int_x^{x_0} \sqrt{\frac{(l-l')^2-1}{4x^2} - \frac{l+l'+1}{2x} + \frac{1}{4}}\, dx, \quad (8)$$

and the value x_0 is determined from the equation:

$$f(x_0) = \frac{1}{4} - \frac{l+l'+1}{2x_0} + \frac{(l-l')^2-1}{4x_0^2} = 0. \quad (9)$$

Hence, disregarding values of the order of n^2/l^2, we have

$$x_0 \approx \frac{(l-l')^2}{2(l+l'+1)}\left(1 + \frac{1}{4}\frac{(l-l')^2}{(l+l'+1)^2}\right). \quad (10)$$

When $x \to x_0$ we obtain

$$z = \frac{2}{3} x_0^{3/2} \sqrt{-f'(x_0)} \left(1 - \frac{x}{x_0}\right)^{3/2} \approx \frac{l-l'}{3}\left(1 - \frac{x}{x_0}\right)^{3/2}. \quad (11)$$

In order to determine the value of constant A, we must find the expression for I (1', 1) in the other extreme case $x \to 0.$ Evaluating integral (8) for $x \to 0,$ we obtain

$$z \approx \frac{1}{2}\left[-(l-l') + (l-l')\ln\frac{(l-l')^2}{x} + \left(l'+\frac{1}{2}\right)\ln\left(l'+\frac{1}{2}\right) - \right.$$
$$\left. -\left(l+\frac{1}{2}\right)\ln\left(l+\frac{1}{2}\right)\right], \quad z' = -\frac{l-l'}{2x}. \qquad (12)$$

Further, taking into account Stirling's formula:

$$n! = \sqrt{2\pi n}\,(n/e)^n\left(1 + O\left(\frac{1}{n}\right)\right), \qquad (13)$$

and also the relationship

$$\underset{z\to\infty}{K_{1/3}(z)} = \sqrt{\frac{\pi}{2z}}\,e^{-z}, \qquad (14)$$

we find

$$I(l', l) \approx A\pi\sqrt{2}\sqrt{\frac{\pi}{l!}\frac{1}{(l-l')!}}\,x^{(l-l')/2}. \qquad (15)$$

Comparing equations (7) and (15), we find that $A = 1/\pi\sqrt{2},$ and therefore in the region of interest to us $(x \to x_0)$ the aymptotic value for the desired function will be equal to

$$I(l', l) = \frac{1}{\pi}\sqrt{\frac{1}{3}\left(1-\frac{x}{x_0}\right)}\,K_{1/3}\left(\frac{l-l'}{3}\left(1-\frac{x}{x_0}\right)^{3/2}\right). \qquad (16)$$

By means of the known recurrent relations between the Sonin - Laguerre polynomials, it is easy to show that

$$
\left.
\begin{aligned}
I(l', l-1) - I(l'-1, l) &= \frac{2\varepsilon}{\pi\sqrt{3}} K_{2/3}\left(\frac{n}{3}\varepsilon^{3/2}\right),\\
I(l', l-1) + I(l'-1, l) &= \frac{2\sqrt{\varepsilon}}{\pi\sqrt{3}} K_{1/3}\left(\frac{n}{3}\varepsilon^{3/2}\right),\\
I(l'-1, l-1)\, I(l'-1, l) + I(l', l)\, I(l', l-1) &= \frac{1}{2}\left(I(l', l-1) + I(l'-1, l)\right)^2,\\
I(l', l) - I(l'-1, l-1) &\sim \frac{n}{l}\left(I(l', l-1) - I(l'-1, l)\right),
\end{aligned}
\right\} \qquad (17)
$$

and, furthermore, disregarding values of the order of n^2/l^2, we have

$$\varepsilon = 1 - \frac{x}{x_0} = (1 - \beta^2 \sin^2\theta)\left(1 + \frac{n}{2l}\right)^2. \qquad (18)$$

Now replacing the sum with respect to n by the corresponding integral in calculating the total radiation intensity, and taking into consideration that almost all the radiation originates in the region $x \to x_0$, we shall have, introducing the new variable $y = (n/3)\varepsilon^{3/2}$:

$$
\begin{aligned}
W = \frac{9e^2\omega_0^2\beta^2}{c\pi^2} \int_0^{\pi} \frac{\sin\theta\, d\theta}{(1-\beta^2\sin^2\theta)^{5/2}} \int_0^{\infty} y^2\, dy \left[1 - \frac{9}{2l}\frac{y}{(1-\beta^2\sin^2\theta)^{3/2}}\right] \times \\
\times \left[K_{2/3}^2(y) + \frac{\cos^2\theta}{1-\beta^2\sin^2\theta} K_{1/3}^2(y)\right]. \qquad (19)
\end{aligned}
$$

1) From (18) it can be seen that the basic term of the expansion $(n/l = 0)$ is a comparatively small value, having when $\theta = \pi/2$ an order of $(1-\beta^2\sin^2\theta) = (mc^2/E)^2$. However, the next term of the expansion $(\sim n/l)$, and also as can be easily shown, higher terms of the expansion (n^2/l^2 etc.,) as well, will contain the same factor, and therefore in the first approximation $(n/l \ll 1)$ we can restrict ourselves merely to terms of the order of n/l.

The latter method of calculating the total radiation intensity, when we shift to the asymptomatic values of the Sonin-Laguerre polynomials up to integration with respect to the angle θ (see /2/), allows us to find the quantum corrections to the total radiation intensity with the assumption that $n/l \ll 1$.

Using further the equations

$$\int_0^\infty K_\nu^2(x)\, x^{\mu-1}\, dx = \frac{2^{\mu-3}}{\Gamma(\mu)} \Gamma^2\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\mu}{2}+\nu\right) \Gamma\left(\frac{\mu}{2}-\nu\right);$$

$$\frac{\beta}{2} \int_0^\pi \frac{\sin\theta\, d\theta}{(1-\beta^2 \sin^2\theta)^{n/2}} \approx \beta \int_0^\infty \frac{dx}{(1-\beta^2+\beta^2 x^2)^{n/2}} = \frac{\Gamma^2\left(\frac{n-1}{2}\right)}{\Gamma(n-1)} \frac{2^{n-3}}{(1-\beta^2)^{(n-1)/2}}, \quad (20)$$

we finally find

$$W = \frac{2}{3} \frac{e^2 \omega_0^2}{c} \left(\frac{E}{mc^2}\right)^4 \left\{1 - \frac{55\sqrt{3}}{16} \left(\frac{\hbar}{mcR}\right) \left(\frac{E}{mc^2}\right)^2 + \ldots\right\}, \quad (21)$$

were R is the orbit radius.

Assuming that $\hbar \to 0$ we obtain the known classical formula for the radiation of relativistic electrons $(\beta \sim 1)$ in magnetic field. Quantum corrections to the total radiation intensity, as well as to the radiation frequency, become apparent only in the energy field /4/:

$$E \sim mc^2 (mcR/\hbar)^{1/3}. \quad (22)$$

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